MATH 5061 Problem Set 5[1]
Due date: Apr 15, 2020

Problems: (Please either type up your assignment or scan a copy of your written assignment into ONE PDF file and send it to me by email on/before the due date. Please remember to write down your name and SID. Questions marked with a † is optional.)

1. Let (M, g) be a Riemannian manifold. Fix $p \in M$.

   (a) Suppose the exponential map $\exp_p$ is defined on the whole tangent space T_pM. Prove that for any $q \in M$, there exists a geodesic γ joining p to q such that $L(\gamma)$ realized the Riemannian distance $\rho(p, q)$ between p and q. Use this to show that (M, ρ) is complete as a metric space.

   (b) Prove the converse of (a), i.e. suppose (M, ρ) is a complete metric space, show that $\exp_p$ is well-defined on T_pM.

2. Let $\gamma(s, t) = \gamma_s(t) : [0, 1] \to M$, $s \in (-\epsilon, \epsilon)$, be a 1-parameter family of smooth curves in a Riemannian manifold (M, g). Denote $V(t) := \frac{\partial}{\partial s}\big|_{s=0} \gamma$ as the variational field along γ_0. Recall that the energy of the curve γ_s is defined by
$$E(\gamma_s) := \int_0^1 \|\gamma_s'(t)\|_g^2 \, dt.$$

   (a) Prove that first variation formula for energy, i.e.
$$\frac{1}{2} \frac{d}{ds}\bigg|_{s=0} E(\gamma_s) = g(\gamma_0'(t), V(t))\big|_{t=0}^{t=1} + \int_0^1 g(D_{\frac{\partial}{\partial t}} \gamma_0'(t), V(t)) \, dt.$$

   (b) If γ_s is a geodesic *for all* s, show that the variation field V is a Jacobi field, i.e. $V(t)$ satisfies the Jacobi field equation:
$$D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} V - R(\gamma_0', V)\gamma_0 = 0.$$

   (c) Prove that every Jacobi field V along γ_0 arises from a 1-parameter family of geodesics as in (b).

3. Let $f : M \to \mathbb{R}$ be a smooth function defined on a Riemannian manifold (M^{m+1}, g). Denote $\Sigma := f^{-1}(a)$ where a is a regular value of f. Show that the mean curvature H, with respect to the unit normal $N = -\frac{\nabla f}{|\nabla f|}$, of the hypersurface Σ is given by $H = \mathrm{div} N$.

4. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^4$ defined by
$$F(u, v) = (\cos u, \sin u, \cos v, \sin v).$$

   (a) Show that F is an isometric immersion (with respect to the flat metrics).

   (b) Prove that the image of F lies inside the round 3-sphere $\mathbb{S}^3 := \{x \in \mathbb{R}^4 \mid |x|^2 = 2\}$, and $\Sigma = F(\mathbb{R}^2)$ is a minimal immersion into $\mathbb{S}^3$, equipped with the induced metric from $\mathbb{R}^4$.

5. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature everywhere. Suppose γ and σ are two geodesics joining the same end points, and γ is homotopic to σ (i.e. there exists a family of curves γ_s, $s \in [0, 1]$, joining the same end points such that $\gamma_0 = \gamma$ and $\gamma_1 = \sigma$). Prove that γ is just a reparametrization of σ.

[1] Last revised on April 15, 2020